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Codimension-two bifurcations and interactions between differently polarised fields for the laser with saturable absorber

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Abstract. The equations for the laser with a saturable absorber in the mean-field limit and exact resonance are considered. For critical values of the number of active atoms in the laser and the absorber the equations exhibit a codimension-two degeneracy of the Takens-Bogdanov type, where a stationary bifurcation and a Hopf bifurcation with infinite period coalesce. In contrast to previous work, the envelopes of the electric field and the atomic polarisations are treated as complex variables. This allows us to take differently polarised laser fields into account. As a consequence the equations possess an O(2) symmetry which is also present in the normal form describing the Takens-Bogdanov bifurcation. We compute the relevant normal form coefficients and apply previous results to divide the space of the physical parameters into a number of regions that give rise to qualitatively different bifurcation scenarios. These include hysteretic and continuous stability exchanges between circularly polarised and modulated linearly polarised laser fields.

1. Introduction

A paradigm for systems with competing instabilities is provided by the laser with a saturable absorber (Arimondo 1985, Degiorgio and Lugiato 1980, Mandel and Erneux 1984, Velarde 1982). While the pumping in the active material acts to destabilise the rest state, the presence of the absorber induces a stabilising effect. The competition between stabilising and destabilising effects gives rise to a variety of phenomena that are not present in ordinary laser systems. Although in the absence of the absorber the first instability, or primary bifurcation, always leads to a steady lasing state which bifurcates supercritically from the rest state, the laser with absorber may exhibit supercritical and subcritical stationary as well as Hopf bifurcations leading to modulated laser fields, depending on the ratios of the various relaxation constants.

An important feature of most systems with competing instabilities is the appearance of a multiple bifurcation of the Takens-Bogdanov type in which a stationary and a Hopf bifurcation with infinite period coalesce (Carr 1981, Guckenheimer and Holmes 1983, Knobloch and Proctor 1981). If no continuous symmetry is present it is determined by a double zero eigenvalue with a 2×2 nilpotent Jordan block of the linearised system, i.e. the flow is contracted to a two-dimensional centre manifold. Generically the Takens-Bogdanov bifurcation has codimension two, although for proper choices of the physical parameters one also encounters higher degeneracies in the equations underlying the laser with absorber. An analysis of these degeneracies is given by Dangelmayr *et al* (1985, 1986) under the assumption that the envelopes of the electric field and the atomic polarisations are real variables. This means that polarisation

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effects of the laser field are neglected. Our purpose here is to study the appearance of the Takens-Bogdanov bifurcation in the equations for the laser with saturable absorber for the case where the envelopes of the electric field and the atomic polarisations are treated as complex variables, confining ourselves to the generic codimension-two situation. The complex treatment introduces an important symmetry into the problem, namely an O(2) symmetry which reflects the fact that the polarisation direction of the laser field is not uniquely determined. As a consequence, the centre manifold is four dimensional, so that the flow on it is described by a system of normal form ODE for four real or two complex variables. A detailed mathematical analysis of the normal form is presented in Dangelmayr and Knobloch (1987) (henceforth referred to as DK). Here we apply these results to the equations for the laser with absorber. In § 2 we compute the non-linear coefficients of the normal form from the basic laser equations. Our main results are presented in § 3, where the space of the physical parameters is divided into a number of regions that give rise to different bifurcation scenarios. They are related to different cases of the normal form according to the classification given in DK. Whereas the real treatment always predicts a stable modulated linearly polarised laser field (Dangelmayr *et al* 1985, 1986), that state turns out to be mostly unstable in favour of a stable circularly polarised field. This and other physical implications are discussed in § 4.

2. Reduction to normal form

In the mean-field limit and exact resonance the laser with saturable absorber is described in dimensionless variables by the equations (Verlarde 1982, Antoranz and Velarde 1988):

$$\dot{a} = \rho[-a + \mathscr{A}p + r_1(1-\mathscr{C})q] \tag{1a}$$

$$\dot{p} = a(1-d) - p \tag{1b}$$

$$\dot{q} = \kappa a(1-e) - r_1 q \tag{1c}$$

$$\dot{d} = \omega[-d + \tfrac{1}{2}(a\bar{p} + \bar{a}p)] \tag{1d}$$

$$\dot{e} = \omega[-r_2 e + \tfrac{1}{2}(a\bar{p} + \bar{a}p)] \tag{1e}$$

where the overbar denotes complex conjugation and the dots represent the time derivative $\mathrm{d}/\mathrm{d}t$. In (1) a is the envelope of the electric field, p and d are, respectively, proportional to the atomic polarisation and the population inversion in the laser, and q, e denote the same variables for the absorber. The parameters ρ, r_1, ω and ωr_2 are the relaxation constants (measured in units of the transverse atomic relaxation rate of the laser) and κ is the ratio of the field-matter coupling constants for the laser and the absorber (Degiorgio and Lugiato 1980). The number of active atoms in the laser and the absorber is proportional to $\mathscr{A}$ and $\mathscr{C}$, respectively, and can be varied by external pumping. The bifurcation behaviour of (1) was investigated in detail by Dangelmayr *et al* (1985, 1986) under the assumption that a, p and q are real variables. Here we consider the general case where a, p and q are complex, thereby taking polarisation effects into account. This introduces an important symmetry that leaves the system (1) invariant. First we can multiply a, p and q by a common phase factor corresponding to the diagonal action of SO(2) in $\mathbb{C}^3$: $(a, p, q) \to \mathrm{e}^{\mathrm{i}\phi}(a, p, q)$. If a, p and q are restricted to real variables then the SO(2) action reduces to the simple sign reversal of (a, p, q)

considered by Dangelmayr *et al* (1985, 1986). The second symmetry is the Z(2) action $(a, p, q) \to (\bar{a}, \bar{p}, \bar{q})$ which also leaves (1) unchanged if the system is augmented by the complex conjugation of the first three equations. Combining these two symmetries leads to the circle group O(2) as the basic symmetry group under which (1) is invariant.

For all values of the parameters the system (1) possesses the trivial solution T: $a = p = q = d = e = 0$. A linear stability analysis along T leads to the following characteristic equation for an eigenvalue l of the linearised system:

$$(l+\omega)(l+r_2\omega)\{l^3+l^2(l+\rho+r_1)+l[r_1(1+\rho)+\rho(1-\mathscr{A})-\kappa\rho r_1(1-\mathscr{C})] \\ +[\rho r_1(1-\mathscr{A})-\kappa\rho r_1(1-\mathscr{C})]\}=0. \tag{2}$$

From (2) we infer that a pitchfork bifurcation (actually a pitchfork of revolution due to the O(2) symmetry) leading to non-trivial steady states takes place if

$$\mathscr{A} = 1-\kappa(1-\mathscr{C}). \tag{3}$$

The steady-state bifurcation degenerates to a codimension-two bifurcation of the Takens–Bogdanov (TB) type if, in addition to (3), the coefficient of l in the characteristic equation vanishes. Combining these two equations gives

$$\mathscr{A} = \mathscr{A}_c \equiv \frac{\rho+r_1}{\rho(1-r_1)} \qquad \mathscr{C} = \mathscr{C}_c \equiv 1+\frac{r_1(1+\rho)}{\kappa\rho(1-r_1)} \tag{4}$$

as conditions for the occurrence of a TB bifurcation which describes the coalescence of a pitchfork and a Hopf bifurcation with infinite period.

If $\mathscr{A}$ and $\mathscr{C}$ are close to their critical values $\mathscr{A}_c$ and $\mathscr{C}_c$ then the flow of (1) is contracted to a two-dimensional complex (or four-dimensional real) centre manifold (Carr 1981, Guckenheimer and Holmes 1983) in virtue of the nilpotent Jordan block corresponding to the zero eigenvalue of the linearised system. Following DK we parametrise the centre manifold by complex variables $(v, w) \in \mathbb{C}^2$ and describe the flow on it by the TB-normal form

$$\dot{v} = w \tag{5a}$$

$$\dot{w} = \mu v + \nu w + [A|v|^2 + B|w|^2 + C(v\bar{w}+\bar{v}w)]v + D|v|^2 w. \tag{5b}$$

In (5), quintic and higher-order terms in (v, w) have been neglected. Details of the reduction of (1) to (5) are presented in the appendix; here we summarise the results. The parameters μ and ν are unfolding parameters which depend on

$$\Delta\mathscr{A} = \mathscr{A} - \mathscr{A}_c \qquad \Delta\mathscr{C} = \mathscr{C} - \mathscr{C}_c$$

and vanish for $\Delta\mathscr{A} = \Delta\mathscr{C} = 0$. To linear order they are given by

$$\mu = -(r_1\rho/l_0)(\Delta\mathscr{A} - \kappa\Delta\mathscr{C}) \tag{6a}$$

$$\nu = (\rho/l_0^2)[(1+\rho)\Delta\mathscr{A} - \kappa r_1(r_1+\rho)\Delta\mathscr{C}] \tag{6b}$$

where

$$l_0 = -(1+r_1+\rho) \tag{7}$$

is a non-degenerate eigenvalue of the linearisation of (1) for $\mathscr{A} = \mathscr{A}_c$ and $\mathscr{C} = \mathscr{C}_c$. The non-linear coefficients A, D and

$$M = 2C + D \tag{8}$$

are given by

$$A=\frac{r_1^3[r_2(r_1+\rho)-\kappa(1+\rho)]}{(1-r_1)r_2l_0} \tag{9a}$$

$$D=\frac{r_1^2(1+\rho)(r_1+\rho)(\kappa r_1-r_2)}{(1-r_1)r_2l_0^2} \tag{9b}$$

$$M=r_1^2M_1/[\omega r_2^2(1-r_1)l_0^2] \tag{9c}$$

where

$$M_1=l_0[\kappa(1+\rho)(2r_1+\omega r_2)-r_1r_2^2(r_1+\rho)(2+\omega)]+3\omega r_2(1+\rho)(r_1+\rho)(\kappa r_1-r_2). \tag{9d}$$

The coefficient B has not been computed explicitly because it plays no role in the qualitative behaviour of (5) (cf DK).

3. Division of the parameter space

In a vicinity of $\Delta\mathscr{A}=\Delta\mathscr{C}=0$ the dynamics of (1) is determined by the normal form (5) which has been analysed in detail in DK. The bifurcations organised by (5) depend crucially on the non-linear coefficients A, D and M. As in the real case (Carr 1981, Guckenheimer and Holmes 1983, Knobloch and Proctor 1981) there are two main cases to be distinguished, namely $A>0$ (subcritical steady lasing) and $A<0$ (supercritical steady lasing). These cases are further distinguished according to the ratio M/D and sgn M (cf figures 3 and 6 in DK). For $A<0$ we encounter eighteen different subcases, denoted by I±, II±, ..., IX±. Here the signs '+' or '−' refer to sgn M and the roman numerals correspond to specific ranges of M/D, defined by: I: $M/D<0$; II: $\infty>M/D>a_8$; III: $a_8>M/D>a_7;\ldots$; IX: $a_2>M/D>a_1$. The numbers $a_j (1\leqslant j\leqslant 8)$ are given by

$$(a_1,\ldots,a_8)=(0,1,\tfrac{5}{4},\tfrac{4}{3},1.35,1.43,\tfrac{5}{3},2). \tag{10a}$$

Analogously, for $A>0$, we encounter eight subcases I±, II±, III± and IV± that occur for $M/D<0$ and $\infty>M/D>a_9$, $a_9>M/D>a_8$, $a_8>M/D>a_1$, respectively, where the signs '±' refer to sgn M as before, and

$$a_9=5. \tag{10b}$$

The task is now to locate the different cases described above in the space of the physical parameters. Confining ourselves to $\kappa=1$, we have to investigate the four-dimensional (r_1, r_2, ρ, ω) space, restricted to positive values of the variables. Since $\mathscr{A}_c$ (equations (4)) is positive we have the further restriction $r_1<1$. This physically admissible domain of the parameter space is divided into two main regions by the equation

$$r_2=r_{2c}\equiv(1+\rho)/(r_1+\rho) \tag{11}$$

corresponding to $A<0$ $(r_2>r_{2c})$ and $A>0$ $(r_2<r_{2c})$. The region $\{A<0\}$ is divided further into subregions by the set of equations

$$M=a_jD \tag{12}$$

where $1\leqslant j\leqslant 8$. Similarly we obtain a division of the region $\{A>0\}$ by $D=0$, i.e. $r_2=r_1$, and by (12) with $j=1,8,9$.

The division of the parameter space is most conveniently discussed in terms of (r_1, r_2) sections for fixed generic values of (ω, ρ). To this end the (ω, ρ) plane is divided into a number of open regions by the curves $\omega = \omega_r(\rho;\ a_j)$, $r = 0, 1, 2$, defined by

$$\begin{aligned} &\omega_0(\rho;\ a_j) = 2/[(4-a_j)(1+\rho)] \qquad 1 \leqslant j \leqslant 8 \\ &\omega_1(\rho;\ a_j) = (4+2\rho)/[1-a_j+\rho(2-a_j)] \qquad 1 \leqslant j \leqslant 7 \\ &\omega_2(\rho;\ a_j) = 2\omega_1(\rho;\ a_j) \qquad 1 \leqslant j \leqslant 7 \end{aligned} \tag{13}$$

as shown in figure 1. If (ω, ρ) is in one of these regions we obtain a certain configuration of curves in the (r_1, r_2) plane which constitute the boundaries between the various cases I±, II± etc. This division of the (r_1, r_2) plane undergoes a qualitative change when (ω, ρ) crosses one of the curves (13). For (ω, ρ) in region 1 of figure 1 $(0 < \omega < \omega_0(\rho;\ a_1))$ we find the division of the (r_1, r_2) plane as sketched in figure 2(a), with the intersection between $\{A = 0\}$ and $\{M = a_jD\}$ given by $r_1 = (2 - a_j/2)\omega(1+\rho)$. All curves $\{M = a_jD\}$ pass through the point $r_1 = r_2 = 1$, although for $2 \leqslant j \leqslant 7$ only the part above

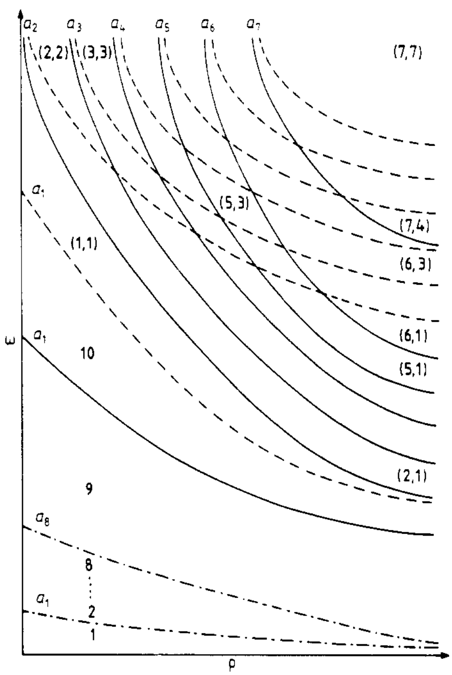

Figure 1. Division of the (ω, ρ) plane by the curves $\omega_0(\rho;\ a_j)$ (chain curves) $\omega_1(\rho;\ a_j)$ (full curves) and $\omega_2(\rho;\ a_j)$ (broken curves), defined by (13). Each curve is labelled by its value a_j. The ω_0 curves separate the regions 1 through 9 as indicated in the figure. Only those corresponding to a_1 and a_8 are shown explicitly. The regions 10 and (i, j) are bounded by the ω_1 and ω_2 curves (see text). The termination points of the chain curves on the ω axis are given by $\omega_{0,j} = 2/(4-a_j)$ yielding $(\omega_{0,1}, \ldots, \omega_{0,8}) = (0.5,\ 0.67,\ 0.73,\ 0.75,\ 0.755,\ 0.78,\ 0.86,\ 1)$. The full and the broken a_1 curves terminate at $\omega = 4$ and $\omega = 8$, respectively. When $\rho \to \infty$ the full curves approach the asymptotes $\omega = \omega_{1,j} = 2/(2-a_j)$ yielding $(\omega_{1,1}, \ldots, \omega_{1,7}) = (1, 2, 2.67, 3, 3.08, 3.51, 6)$, whereas the broken curves approach $\omega = 2\omega_{1,j}$. When $\omega \to \infty$ both the full and the broken curves approach, for $2 \leqslant j \leqslant 7$, the values $\rho_j = (a_j - 1)/(2-a_j)$ yielding $(\rho_2, \ldots, \rho_7) = (0,\ 0.34,\ 0.5,\ 0.54,\ 0.75,\ 2)$. No attempts are made to preserve scales.

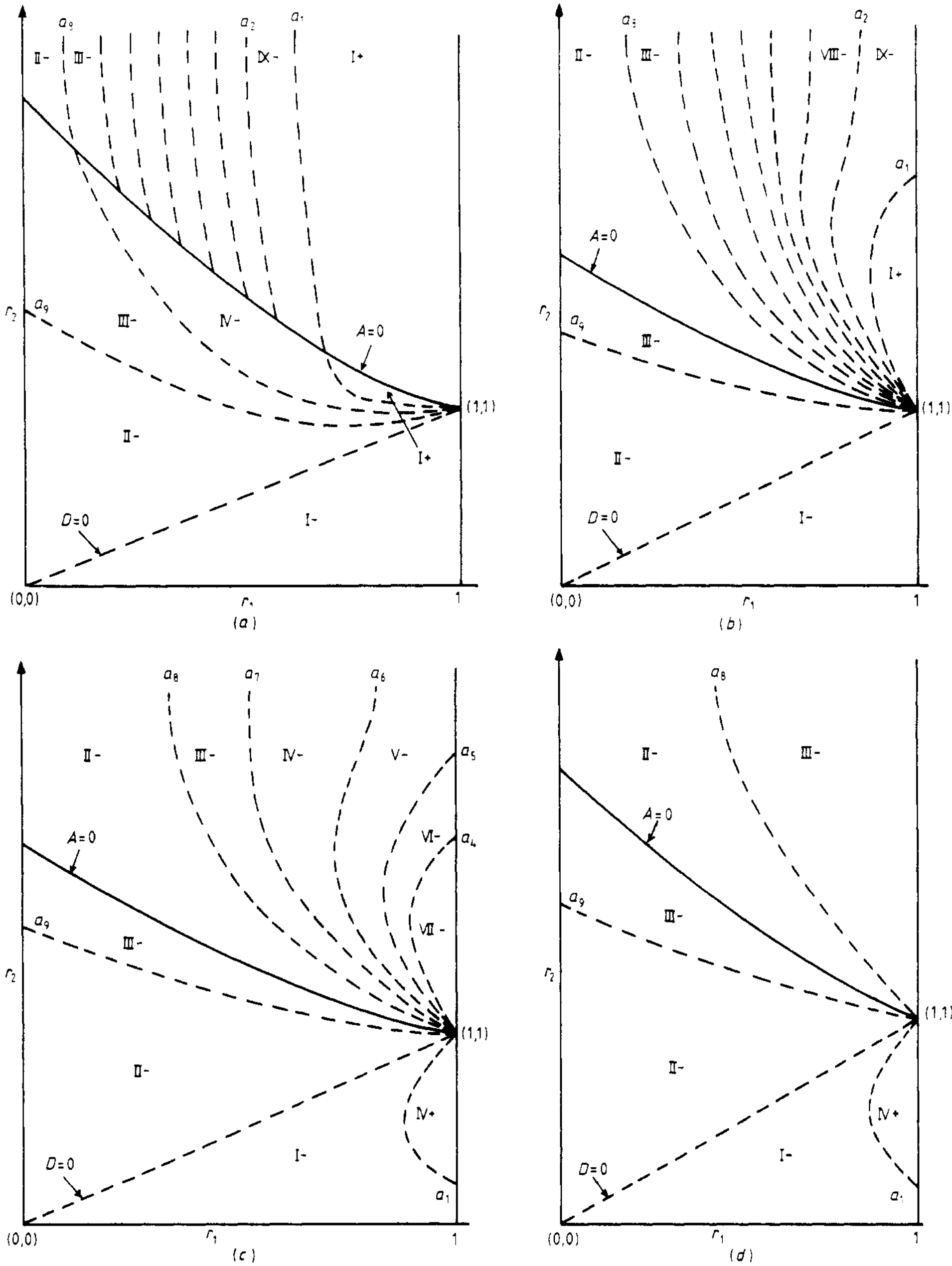

Figure 2. The division of the (r_1, r_2) plane for (ω, ρ) in regions 1, 10, (5, 3) and (7, 7) of figure 1 is sketched in (*a*), (*b*), (*c*) and (*d*), respectively. The full curve is defined by $r_2 = r_{2c}$ (equation (11)) and distinguishes the main sign cases $A<0$ $(r_2 > r_{2c})$ and $A>0$ $(r_2 < r_{2c})$. It terminates on the r_2 axis at $r_2 = (1+\rho)/\rho$. The broken curves labelled by $a_j (1 \leq j \leq 9)$ correspond to $M = a_j D$. For $2 \leq j \leq 7$ they are restricted to the domain $\{A<0\}$ and for $j=9$ to $\{A>0\}$. The latter terminates on the r_1 axis at $r_2 = (1+\rho)/2\rho$. The other intersection and termination points as well as the asymptotes depend on (ω, ρ) and are given in the text. The line $D \doteq 0$ is given by $r_1 = r_2$. Each of the main domains $\{A<0\}$ and $\{A>0\}$ is divided by the broken curves into a number of regions associated with one of the cases I±, II±, ... of normal form (5) as marked in the diagrams. No attempts are made to preserve scales.

$r_2 = r_{2c}$ is relevant so that these boundaries terminate on $\{A = 0\}$. When (ω, ρ) approaches the curve $\omega = \omega_0(\rho; a_1)$ from below, the intersection of $\{M = a_1 D\}$ with $\{A = 0\}$ in figure 2(a) moves towards $(r_1, r_2) = (1, 1)$ and remains there for $\omega \geqslant \omega_0(\rho; a_1)$, thus leading to the disappearance of region I+ in $\{A > 0\}$. Similarly, for (ω, ρ) approaching successively the curves $\omega = \omega_0(\rho; a_j)$ $(j = 2, 3, \ldots, 7)$, the termination points of $\{M = a_j D\}$ on $\{A = 0\}$ move towards $(r_1, r_2) = (1, 1)$ so that in region 8 of figure 1 the boundaries $\{M = a_j D\}$ in the (r_1, r_2) plane emerge from $(r_1, r_2) = (1, 1)$ for $1 \leqslant j \leqslant 7$ and extend to $r_2 = \infty$ within the region $\{A < 0\}$. The same event takes place for the boundary $\{M = a_8 D\}$ when (ω, ρ) approaches $\omega = \omega_0(\rho; a_8)$, i.e. for (ω, ρ) in region 9 of figure 1 the region IV− in $\{A > 0\}$ has disappeared.

When $r_2 \to \infty$, the curves $\{M = a_j D\}$ tend to limiting values $r_1 = r_{1,j} > 0$ which are determined by the quadratic equations

$$r_1^2(2+\omega) + r_1(1+\rho)(2+\omega) - (3 - a_j)\omega(1+\rho) = 0.$$

For $1 \leqslant j \leqslant 7$ these limiting values satisfy $0 < r_{1,j} < 1$ if $\omega < \omega_1(\rho; a_j)$ and $r_{1,j} > 1$ if $\omega > \omega_1(\rho; a_j)$, whereas $0 < r_{1,8} < 1$ for all values of (ω, ρ). Consequently, when $\omega = \omega_1(\rho; a_j)$ (full curves in figure 1) is approached from below, the boundary $\{M = a_j D\}$ develops for $2 \leqslant j \leqslant 7$ a termination point $(r_1, r_2) = (1, r_{2,j})$ on the boundary $\{r_1 = 1\}$ of the physically admissible domain (figures 2(b, c)). The r_2 coordinate of the termination point is given by

$$r_{2,j} = 2(2+\rho)/\{\omega[1 - a_j + \rho(2 - a_j)] - 2(2+\rho)\}$$

and satisfies $r_{2,j} > 1$ for $\omega_1(\rho; a_j) < \omega < \omega_2(\rho; a_j)$. Thus in region 10 of figure 1, defined by $\omega_1(\rho; a_1) < \omega_2(\rho; a_1)$, we obtain the division of the (r_1, r_2) plane as sketched in figure 2(b). If $\omega > \omega_2(\rho; a_j)$ we have $0 < r_{2,j} < 1$ which implies that the boundary $\{M = a_j D\}$ disappears for $2 \leqslant j \leqslant 7$, whereas in the case $j = 1$ it is located entirely in $\{A > 0\}$ and leads here to the appearance of region IV+ (figures 2(c, d)).

We define the regions (i, j) for $1 \leqslant i \leqslant 6$, $1 \leqslant j \leqslant i$ and $i = 7$, $j = 4, 5, 6, 7$ in the (ω, ρ) plane by (cf figure 1)

$$(i, j) = \{\omega_1(\rho; a_i) < \omega < \omega_1(\rho; a_{i+1}),\ \omega_2(\rho; a_j) < \omega < \omega_2(\rho; a_{j+1})\}$$

where we have set $\omega_1(\rho; a_8) = \omega_2(\rho; a_8) = \infty$. In these regions the boundaries $\{M = a_k D\}$ with $8 \geqslant k > i$ extend in $\{A < 0\}$ towards $r_2 = \infty$ (i.e. $0 < r_{1,k} < 1$), whereas those with $2 \leqslant k \leqslant j$ have disappeared and those with $j < k \leqslant i$ terminate on $\{r_1 = 1\}$. To illustrate this behaviour we have sketched the division of the (r_1, r_2) plane for (ω, ρ) in regions $(5, 3)$ and $(7, 7)$ in figures 2(c, d), respectively. This completes our discussion of the division of the parameter space.

4. Discussion and conclusion

From figures 1 and 2 we conclude that the cases $A < 0$: I+, II− through IX−, and $A > 0$: I±, II−, III−, IV− of the normal form (5) appear in the laser with a saturable absorber for $\kappa = 1$. Stability and bifurcation diagrams corresponding to all cases of the normal form are summarised in § 7 of DK. Some of them are also discussed by Peplowski and Haken (1988). We will not present all bifurcation diagrams that are relevant for the physical system, but confine ourselves to those which occur for all values of (ω, ρ). These are the cases $A > 0$: I−, II−, III− and $A < 0$: II−, III−. The ensuing bifurcation diagrams are shown in figure 3(a, b). The notations SS, TW, SW

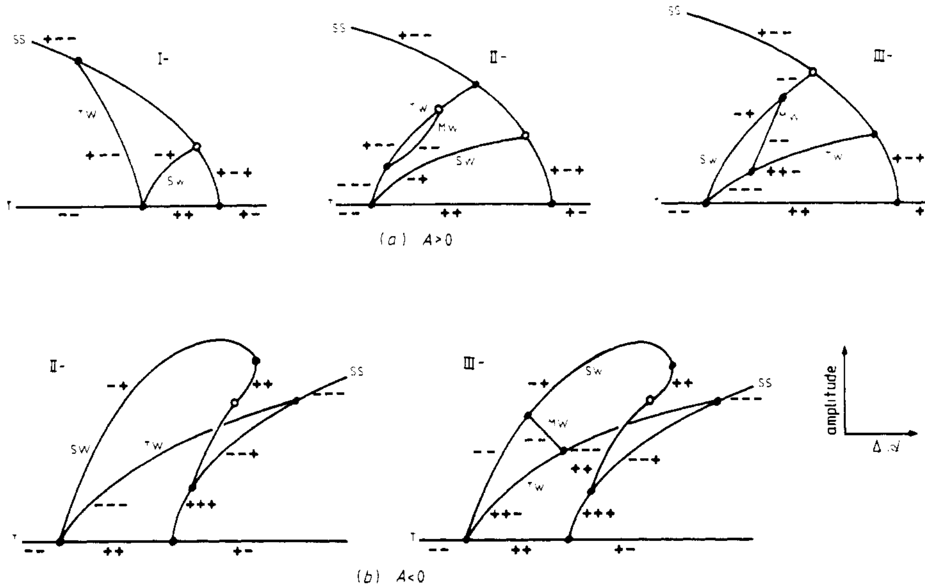

Figure 3. Bifurcation diagrams for some of the cases of normal form (5) that occur in the laser problem. They show appropriate amplitudes of the various solutions of (5) (T, SS, SW, TW, MW) plotted against $\Delta\mathscr{A}$. The meanings of the solutions are explained in the text. Ordinary bifurcations are shown as dots and homoclinic or heteroclinic bifurcations as small circles (see DK for a more detailed discussion of the nature of the various bifurcation points). The signs represent stability symbols along the branches: '−' refers to negative and '+' to positive real parts of the relevant eigenvalues.

and MW, which are used in figure 3 to denote the solutions of (5), are adopted from DK. They are abbreviations for steady states, travelling waves, standing waves and modulated waves, respectively. This notation has its origin in a wave context, where $v\exp(ikx)$ represents a wave field in an infinitely extended system with x being the space variable and k the wavenumber (see, e.g., Dangelmayr and Knobloch 1986). The solution T represents the rest state $v=\omega=0$, i.e. the spatially and temporally homogeneous solution.

For the laser system considered in this paper the meaning of the solutions of (5) is different. Here, a represents the envelope of the electric field E, i.e. $E=a(t)\exp[\mathrm{i}(k_0x-\omega_0\tau)]$, where τ is a fast and t a slow time variable and ω_0 and k_0 are the basic frequency and wavenumber of the laser. Because a is proportional to v on the centre manifold, we may regard v as the complex amplitude of the lasing field. Along the SS branch the magnitude and the direction of v is constant so that this solution corresponds to a linearly polarised laser field. By virtue of the O(2) symmetry the polarisation direction is not unique and, therefore, will be determined by fluctuations. On a TW branch v has the form $v=v_0\exp(\mathrm{i}\Omega t)$ where Ω and $|v_0|$ are determined by ν and μ. Consequently, TW represents a circularly polarised laser field. The SW and MW solutions correspond to modulations of the linearly and circularly polarised fields, respectively. In particular, close to a homoclinic or heteroclinic bifurcation we expect them to be observed as pulsed modes with sharply defined spikes (Q-switch solutions, see Arimondo 1985).

An important consequence following from our analysis is that a supercritical TW always possesses a stable sub-branch, whereas SW is always unstable if no MW exists.

A stable SW branch occurs only when a MW is also present, and then a stability exchange between TW and SW takes place via MW. This stability exchange may be continuous (stable MW) as in the cases III− for $A<0$ and $A>0$ (figure 3) or hysteretic (unstable MW) as in the cases $A<0$: IV− through IX− (see figure 8 in DK). On the other hand, if (5) were analysed with (v, w) restricted to real values, TW and MW would not be present whereas SW would always appear to be stable (Carr 1981, Guckenheimer and Holmes 1983, Knobloch and Proctor 1981). Thus the real treatment can lead to erroneous results. We mention here an investigation of Elgin and Molina Garza (1987) who studied a related problem, corresponding to the case $r_1 = 0$ of the present paper. These authors obtained a supercritical Hopf bifurcation for some negative value of ρ. However, since they confined themselves to real variables, they found a stable SW and no TW. We expect that a complex treatment will reveal an unstable SW in favour of a stable TW. Corrections of this type are also expected when the analyses of Dangelmayr *et al* (1985, 1986) are extended to the complex case.

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Appendix

In order to derive the TB-normal form (5) from the basic equations (1) we begin with the linear transformation $(a, p, q) \to (V, W, Z)$, defined by

$$\begin{aligned} a &= r_1 V + (1+\rho)(r_1+\rho)Z \\ p &= r_1(V-W) - (1+\rho)Z \\ q &= \kappa V - (\kappa/r_1) W - \kappa(r_1+\rho)Z. \end{aligned} \tag{A1}$$

The resulting system of ODE for (V, W, Z, d, e) has the form

$$\begin{aligned} \dot{V} &= (r_1 s_3 - l_o)\Theta + W - (s_2 s_3 / r_1 s_1 l_o^2)\, Y(d - r_1^2 e) \\ \dot{W} &= -r_1 l_o \Theta + (1/s_1 l_o)\, Y(s_3 d - r_1 s_2 e) \\ \dot{Z} &= r_1 \Theta + l_o Z + (1/s_1 l_o^2)\, Y(d - r_1^2 e) \\ \dot{d} &= -\omega d + \omega[r_1^2 |V|^2 - s_2^2 s_3 |Z|^2 - r_1^2 X_1 + r_1 s_2(s_3 - 1)X_2 - r_1 s_2 s_3 X_3] \\ \dot{e} &= -r_2 \omega e + \omega\kappa[r_1 |V|^2 - s_2 s_3^2 |Z|^2 - X_1 + s_3(s_2 - r_1)X_2 - (s_2 s_3/r_1)X_3] \end{aligned} \tag{A2}$$

where

$$s_1 = 1 - r_1 \qquad s_2 = 1+\rho \qquad s_3 = r_1 + \rho \tag{A3}$$

$$Y = r_1 V + s_2 s_3 Z \qquad X_1 = \mathrm{Re}(VW) \qquad X_2 = \mathrm{Re}(VZ) \qquad X_3 = \mathrm{Re}(WZ) \tag{A4}$$

and

$$\Theta = \vartheta_1 V - \vartheta_2 W - \vartheta_3 Z \tag{A5}$$

with

$$\begin{aligned}\vartheta_1 &= (\rho/l_o^2)(\Delta\mathscr{A} - \kappa\Delta\mathscr{C})\\ \vartheta_2 &= (\rho/r_1 l_o^2)(r_1\Delta\mathscr{A} - \kappa\Delta\mathscr{C})\\ \vartheta_3 &= (\rho/r_1 l_o^2)(s_2\Delta\mathscr{A} - \kappa r_1 s_3\Delta\mathscr{C})\end{aligned} \tag{A6}$$

i.e. the linearisation of (A2) is in Jordan-normal form for $\Delta\mathscr{A} = \Delta\mathscr{C} = 0$. Approximating the centre manifold of (A2) by

$$\begin{aligned}Z &= 0\\ d &= r_1^2|V|^2 + (r_1/\omega)^2(2+\omega)(|W|^2 - \omega X_1)\\ e &= (\kappa r_1/r_2)|V|^2 + (\kappa/r_2^3\omega^2)(2r_1+\omega r_2)(|W|^2 - r_2\omega X_1)\end{aligned} \tag{A7}$$

yields the reduced system

$$\begin{aligned}\dot V &= (r_1 s_3 - l_o)(\vartheta_1 V - \vartheta_2 W) + W + [a_1|V|^2 + b_1|W|^2 + c_1(V\bar W + \bar V W)]V\\ \dot W &= -r_1 l_o(\vartheta_1 V - \vartheta_2 W) + [a_3|V|^2 + b_3|W|^2 + c_3(V\bar W + \bar V W)]V\end{aligned} \tag{A8}$$

where the same notation as in DK (equation (2.5)) has been used for the cubic coefficients a_1, b_1, etc. In terms of the physical parameters they are given by

$$\begin{aligned}a_1 &= (r_1^2 s_2 s_3/s_1 r_2 l_o^2)(\kappa r_1 - r_2)\\ b_1 &= (r_1^2 s_2 s_3/s_1\omega^2 r_2^3 l_o^2)[\kappa(2r_1+\omega r_2) - r_2^3(2+\omega)]\\ c_1 &= (r_1^2 s_2 s_3/2\omega s_1 r_2^2 l_o^2)[r_2^2(2+\omega) - \kappa(2r_1+\omega r_2)]\\ a_3 &= (r_1^3/s_1 r_2 l_o)(r_2 s_3 - \kappa s_2)\\ b_3 &= (r_1^2/s_1\omega^2 r_2^3 l_o)[s_3 r_1 r_2^3(2+\omega) - s_2\kappa(2r_1+\omega r_2)]\\ c_3 &= (r_1^2/2\omega s_1 r_2^2 l_o)[s_2\kappa(2r_1+\omega r_2) - r_1 r_2^2 s_3(2+\omega)].\end{aligned} \tag{A9}$$

By means of the near-identity transformation

$$\begin{aligned}v &= V - \tfrac{1}{2}c_1|V|^2 V - \tfrac{1}{2}b_1 V^2\bar W\\ w &= W + (r_1 s_3 - l_o)(\vartheta_1 V - \vartheta_2 W) + a_1|V|^2 V + \tfrac{1}{2}c_1 V^2\bar W\end{aligned} \tag{A10}$$

we obtain from (A8) the normal form (5) with

$$\mu = -r_1 l_o\vartheta_1 \qquad \nu = (r_1 s_3 - l_0)\vartheta_1 + r_1 l_0\vartheta_2 \tag{A11}$$

and

$$A = a_3 \qquad B = b_3 + c_1 \qquad C = a_1 + c_3 \qquad D = a_1. \tag{A12}$$

The expressions (6) and (9*a*–*d*) follow from (A6), (A11) and (A9), (A12), respectively.

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